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Capillary–gravity waves of solitary type and envelope solitons on deep water

By MICHAEL S. LONGUET-HIGGINS

Institute for Nonlinear Science, University of California, San Diego, La Jolla, CA 92093-0402, USA

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The existence of steady solitary waves on deep water was suggested on physical grounds by Longuet-Higgins (1988) and later confirmed by numerical computation (Longuet-Higgins 1989; Vanden-Broeck & Dias 1992). Their numerical methods are accurate only for waves of finite amplitude. In this paper we show that solitary capillary–gravity waves of small amplitude are in fact a special case of envelope solitons, namely those having a carrier wave of length $2\pi(T/\rho g)^{\frac{1}{2}}$ (g = gravity, T = surface tension, ρ = density). The dispersion relation $c^2 = 2(1-\frac{11}{32}\alpha^2_{\max})$ between the speed c and the maximum surface slope $\alpha_{\max}$ is derived from the nonlinear Schrödinger equation for deep-water solitons (Djordjevik & Redekopp 1977) and is found to provide a good asymptote for the numerical calculations.

1. Introduction

The theoretical existence of a steady capillary–gravity wave of solitary type on deep water was first suggested on physical grounds by Longuet-Higgins (1988) and was confirmed by accurate numerical computation in a second paper (Longuet-Higgins 1989, to be referred to as LH2). Experimentally, waves with a profile very similar to those predicted have been observed in laboratory experiments by Zhang & Cox (1993). Meanwhile, Vanden-Broeck & Dias (1992, to be referred to as VBD) have published numerical calculations of both forced and free capillary–gravity waves, using a different numerical technique. They have verified the existence of the 'depression' solitary waves found in LH2, which have a wave trough in the plane of symmetry, and have found also 'elevation' solitary waves which have a crest in the plane of symmetry. It is to be noted that in both types of wave the total added mass must be zero, by a theorem proved in LH2. This of course contrasts with the situation in shallow water, or in any finite depth.

The surface profiles calculated in VBD suggest that as the amplitude of the solitary waves is decreased, so the waves spread out horizontally and develop more and more undulations. In this respect they come to resemble 'envelope solitons' on deep water. The theory of deep-water solitons, originally developed for pure gravity waves by Benny & Newell (1967), Hasimoto & Ono (1972) and Zakharov & Shabat (1972) was first discussed for capillary–gravity waves by Djordjevik & Redekopp (1977) and later by Ablowitz & Segur (1979). In general, these solutions are time-dependent. A carrier wave, of slowly varying amplitude and phase, progresses through the group with its own phase speed, while the wave envelope, described by a complex amplitude function, progresses unchanged with the corresponding group velocity. For pure gravity waves in deep water, the two velocities are always unequal. However, for general capillary–gravity waves there is one special wavenumber at which the phase speed equals the group speed, according to linear theory (Lamb 1932, Chap. 9, and figure 3 below). For waves in the neighbourhood of this wavenumber one suspects the existence of *steady* envelope solitons.

The theory of envelope solitons given by Djordjevik & Redekopp (1977), for example, applies only to solitons of sufficiently small amplitude. The purpose of this note is to enquire whether the particular *steady* solitons of this family are the small-amplitude extension of the finite-amplitude solitary waves calculated in LH2 and VBD.

A crucial test is the amplitude–dispersion relation, that is the relation between the phase speed c and the wave amplitude B, or the maximum angle of inclination $\alpha_{\max}$ of the free surface in the wave motion. This relation was shown in figure 5 and table 1 of LH2 for waves in the range $0.926 < c < 1.30$. The calculated values have been replotted in figures 1 and 2 below. Also plotted are points derived from the computations given in VBD. In the same diagrams we show the small-amplitude asymptote derived from envelope soliton theory (§3 below), and it will be seen that the agreement is convincing.

In §2 of this paper we give definitions and review briefly the numerical methods and results of LH2 and VBD. In §3 we use the theory of small-amplitude envelope solitons to derive a dispersion relation for solitary waves of small amplitude. A discussion follows in §4. The relation of our results to some very recent work by Dias & Iooss (1993) is described in an Appendix.

2. Computations of solitary waves; definitions

We consider steady, irrotational waves in an ideal, incompressible fluid of infinite depth. If we take axes $(0x, 0y)$ moving horizontally with the phase-speed c, the flow appears independent of the time. We may take the origin 0 in the mean level and the y-axis increasing vertically upwards. If ϕ and ψ denote the velocity potential and stream function, then $\phi \to cx$ at infinity.

The velocity components (u, v) can be expressed as

$$u - \mathrm{i}v = c\,\mathrm{e}^{\beta - \mathrm{i}\alpha} \tag{2.1}$$

where α is the inclination and $c\,\mathrm{e}^{\beta}$ the magnitude of the velocity vector. At infinity, both α and β tend to zero. Moreover, since $(\alpha + \mathrm{i}\beta)$ is an analytic function of $(u - \mathrm{i}v)$, which in turn is an analytic function of $(\phi + \mathrm{i}\psi)$, it follows that α and β are conjugate functions of ϕ and ψ satisfying the Cauchy–Riemann relations.

In LH2, we took units of mass, length and time so as to make the density ρ, the surface tension T and acceleration due to gravity g satisfy

$$\rho = 1, \quad T = 1, \quad g = 1. \tag{2.2}$$

The condition of constant pressure on the free surface ($\psi = 0$) is then given by

$$\sin\alpha + c^3\,\mathrm{e}^{3\beta}\beta_\phi - c^2\,\mathrm{e}^{2\beta}(\alpha_\phi\beta_\phi + \alpha_{\phi\phi}) = 0; \tag{2.3}$$

see LH2, §2. By the transformation

$$w = \phi + \mathrm{i}\psi = \mathrm{i}b\frac{\zeta + 1}{\zeta - 1} \tag{2.4}$$

(b a constant) the half-plane $\psi < 0$ was mapped onto the interior of the unit circle $|\zeta| = 1$. The function

$$G(w) = (w - a)^2(\alpha + \mathrm{i}\beta) \tag{2.5}$$

was then expanded in a power series in ζ, and the coefficients were determined

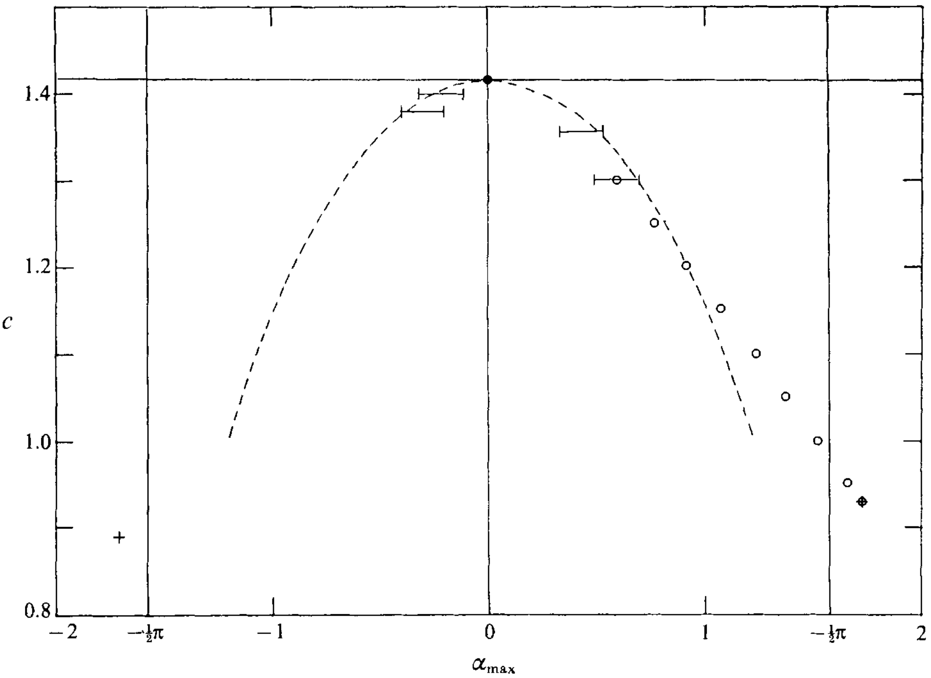

FIGURE 1. The phase speed of solitary waves shown as a function of α_{max}, the maximum angle of inclination of the free surface. Open circles are from LH2, error bars are from figures 6 and 7 of VBD and the dashed curve is the asymptote (3.20).

numerically by satisfying the boundary condition at points around the circle. The solutions converged numerically when $0.9267 \leqslant c < 1.3$, see figure 1. The lower value of c corresponded to waves of limiting amplitude in which the surface enclosed a 'bubble of air'.

In VBD, Vanden-Broeck & Dias adopted a different approach, solving in integro-differential system of equations for x and y in terms of ϕ and ψ, to be satisfied on $\psi = 0$. In general, their formulation included a prescribed pressure distribution $\epsilon P(\phi)$ applied at the free surface. In the special case $\epsilon = 0$ they found free solitary waves of both depression and elevation. The former confirmed the calculations in LH2 and extended them to larger values of c (lower values of α_{max}).

As parameters for the solitary waves, VBD used the difference in level Δy between the free surface in the plane of symmetry $x = 0$ and at $x = \infty$. It was found that Δy was positive or negative for waves of depression or elevation, respectively. However, as appears from their figure 3, Δy is not monotonic throughout the range of c. In contrast, the parameter α_{max} is monotonic and smoothly varying. We can allow α_{max} to be positive for waves of depression and negative for waves of elevation.

Figure 1 shows the phase speed c plotted against α_{max}. The open circles are taken from table 1 of LH2. The plots with error bars have been measured graphically from figures 6 and 7 of VBD, which correspond to waves of elevation and depression respectively. The crosses correspond to the limiting waves. Here, the maximum angle of inclination is known quite accurately from the limiting form of the 'bubbles'; see Longuet-Higgins (1988).

Note that VBD choose units of length and time so that $T = 1$ and $c = 1$ always. The phase speed c in our units can be identified with $\alpha^{-\frac{1}{4}}$, where $\alpha = gT/\rho U^4$ is the dimensionless parameter defined in their equation (2.5).

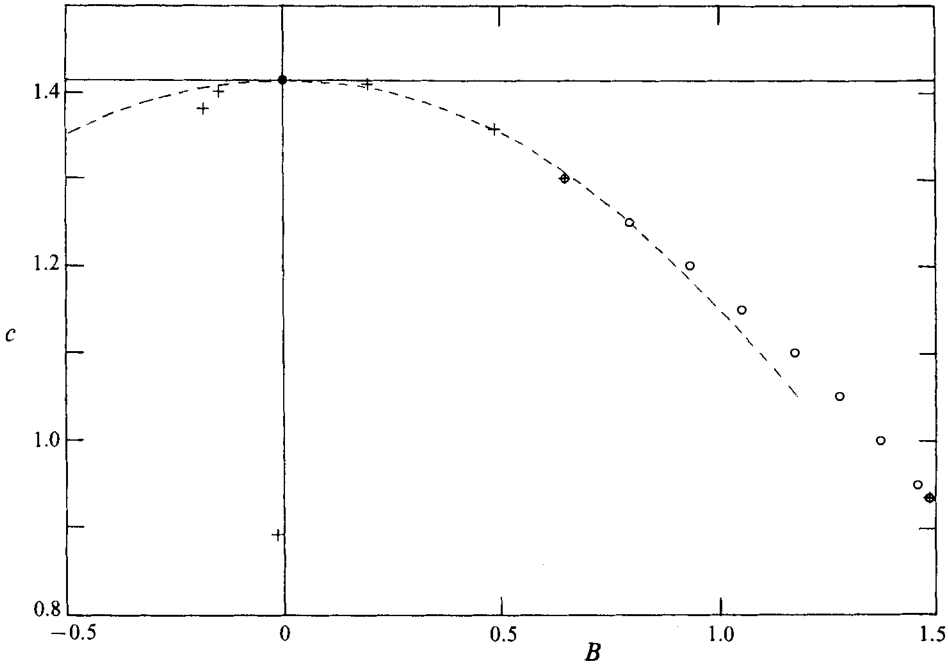

FIGURE 2. The phase speed of solitary waves shown as a function of the Bernoulli constant B (B = minus surface displacement at $x = 0$). Open circles are from LH2, crosses are from table 1 of VBD and the dashed curve is the asymptote (3.20).

Figure 2 shows c plotted against the Bernoulli constant

$$B = g(y - y_0) + \tfrac{1}{2}(q^2 - c^2) + T\kappa/\rho$$

defined in LH2, equation (4.1). Note that at $x = \infty$, the particle speed q equals c, and κ vanishes, so that $B = g(y_\infty - y_0)$. VBD define $A = (y_\infty - y_0)(T/\rho U^2)$, so that we may identify B with $-Ac^2$. The circles in figure 2 again correspond to the values in table 1 of LH2. The crosses correspond to figures 6 and 7 of VBD but are taken from the more accurate values given in their table 1. It will be seen that the points corresponding to $c = 0.927$ and $c = 1.30$ from VBD agree precisely with the corresponding points from LH2, as was noted by Vanden-Broeck & Dias, confirming the identity of the depression waves.

For lower values of $|\alpha_{\max}|$ the surface profiles shown in VBD tend to resemble *envelope* solitons, with an increasingly large number of waves in a group. For this reason, their method of computation also fails at low amplitudes. Their parameter α appears to approach the value $\frac{1}{4}$, corresponding to a phase speed $c = \surd 2$, in our units. Vanden-Broeck & Dias note that when $|c| < \surd 2$ the linear dispersion relation

$$k^2 - c^2 k + 1 = 0 \tag{2.6}$$

(k = wavenumber, in our units) has complex roots

$$k = \tfrac{1}{2}[c^2 \pm \mathrm{i}(4 - c^4)^{\frac{1}{2}}] \tag{2.7}$$

suggesting the existence of waves which decay exponentially towards infinity, as in an envelope soliton. However, they do not derive any analytical relation between the speed of the soliton and its shape or amplitude. This we shall now do.

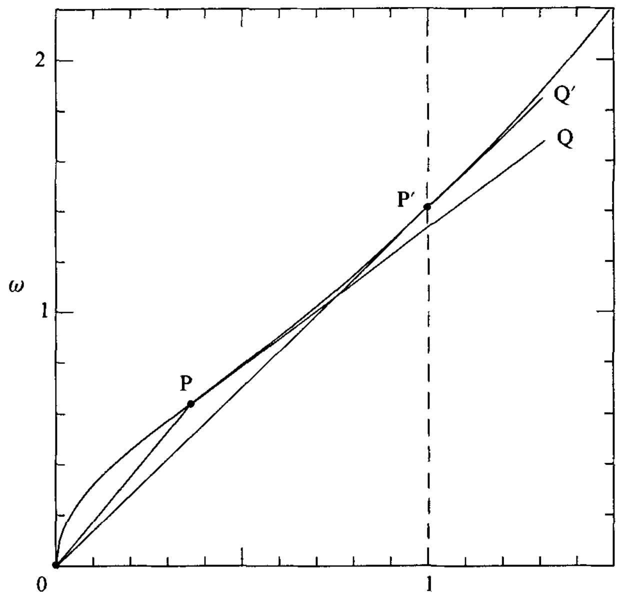

FIGURE 3. The dispersion relation (3.1) for linear capillary–gravity waves in deep water. The phase-velocity c and group velocity c_g at a typical point P are given by the gradients of 0P and PQ respectively. P' corresponds to the minimum phase speed c_{max}. When P lies at P′, then the gradients of 0P and PQ are equal, so $c = c_g$.

3. Envelope solitons

The simplest way to write the linear dispersion relation (2.6) is in the form

$$\omega^2 = k + k^3, \tag{3.1}$$

where ω and k are the radian frequency and the wavenumber respectively, see figure 3. Then the phase speed c and group velocity c_g are given by ω/k and $d\omega/dk$ respectively, and it is obvious from figure 3 that c and c_g are equal only at the phase speed minimum, which occurs when

$$k = 1, \quad c = c_g = \sqrt{2}. \tag{3.2}$$

In their theory of envelope solitons, Djordjevik & Redekopp (1977) assume a general expression for the surface elevation η in a wave train of small slope and slowly varying amplitude which we may write in the form

$$\eta = \frac{2\epsilon\omega}{g(1+\tilde{T})} \operatorname{Re}\{A\, e^{i(kx+\omega t)}\}, \tag{3.3}$$

where ϵ is a small parameter and

$$\tilde{T} = k^2 T/\rho g. \tag{3.4}$$

The complex amplitude A is a function of the slow variable $\xi = \epsilon(x + c_g t)$ and the slow time $\tau = \epsilon t$. They show that in deep water, $A(\xi, \tau)$ must satisfy the nonlinear Schrödinger equation

$$i\omega A_\tau = \lambda A_{\xi\xi} + \mu |A|^2 A \tag{3.5}$$

correct to order ϵ^3, where

$$\lambda = \frac{g}{8k}\frac{3\tilde{T}^2 + 6\tilde{T} - 1}{\tilde{T}+1}, \qquad \mu = \frac{k^4}{4}\frac{2\tilde{T}^2 + \tilde{T} + 8}{(\tilde{T}+1)(2\tilde{T}-1)}. \tag{3.6}$$

In the special case when $k = 1$ in our units, then $\omega = \surd 2$ and $\tilde{T} = 1$, making

$$\lambda = \tfrac{1}{2}, \qquad \mu = \tfrac{11}{8}. \tag{3.7}$$

The solution of (3.5) is found by writing

$$A(\xi, \tau) = \rho\, \mathrm{e}^{\mathrm{i}(\phi - \gamma\tau)}, \tag{3.8}$$

where ρ and ϕ are functions of ξ only. Then from the real and imaginary parts of (3.5) we get

$$\lambda(\rho_{\xi\xi} - \rho\phi_\xi^2) = \omega\gamma\rho - \mu\rho^3 \tag{3.9}$$

and

$$2\rho_\xi\phi_\xi + \rho^2\phi_{\xi\xi} = 0 \tag{3.10}$$

respectively. On integrating (3.10) we find

$$\rho^2\phi_\xi = \text{constant}, \tag{3.11}$$

which must vanish since we require $\rho \to 0$ and ϕ_ξ bounded as $|\xi| \to \infty$. Thus ϕ is a constant, and symmetry about $\xi = 0$ requires that $\phi = 0$ or π. Equation (3.9) can now be integrated to give

$$\lambda\rho_\xi^2 = \omega\gamma\rho^2 - \tfrac{1}{2}\mu\rho^4, \tag{3.12}$$

the constant of integration again being 0 for a solitary wave. The unique solution of (3.12) tending to zero as $|\xi| \to \infty$ is

$$\rho = a \operatorname{sech} b\xi \tag{3.13}$$

provided

$$a^2 = 2\omega\gamma/\mu, \quad b^2 = \omega\gamma/\lambda. \tag{3.14}$$

So from (3.3), (3.8) and (3.13) we have

$$\eta = \pm M\, \mathrm{e}^{\mathrm{i}(x + \omega t - \epsilon^2\gamma t)}, \tag{3.15}$$

where

$$M = \epsilon\omega a = 2^{\frac{5}{4}}(\gamma/\mu)^{\frac{1}{2}}. \tag{3.16}$$

Equation (3.15) represents a wave whose (negative) phase speed has been reduced in magnitude by the amount

$$\Delta c = \epsilon^2\gamma = 2^{-\frac{5}{2}}\mu M^2, \tag{3.17}$$

a second-order quantity. There will be a corresponding reduction in the group velocity c_g. However, the replacement of c_g by c_g', say, in the expression $\xi = \epsilon(x - c_g t)$ will bring about only a higher-order change in the differential equation (3.5) for A, so that to lowest order (3.5) is still valid. Hence, the solitary-wave solution of the modified equation indeed represents a steady, progressive wave to this order.

The dispersion relation for this wave is found from (3.17). It is convenient to write this in terms of the maximum surface slope

$$\alpha_{\max} = Mk. \tag{3.18}$$

Thus in dimensionless units, since $\mu = \frac{11}{8}$, we find

$$c = \pm\surd 2(1 - \tfrac{11}{64}\alpha_{\max}^2). \tag{3.19}$$

or to the same degree of approximation

$$c^2 = 2(1 - \tfrac{11}{32}\alpha_{\max}^2). \tag{3.20}$$

This asymptote is represented by the dashed curve in figure 1. Since in our units the wave number is unity, we can also replace $\alpha_{\max}$ by a, the wave amplitude at the origin.

This is equivalent to the Bernoulli constant B. The corresponding asymptote is shown in figure 2.

4. Discussion

In the more accurate figure 2 it will be seen that the numerically determined values for solitary waves of finite amplitude approach quite well the asymptotic curve for envelope solitons of infinitesimal amplitude, at least for the waves of depression ($B > 0$). For waves of elevation, on the other hand, the calculated points soon diverge much further from the asymptote. This we should expect, since the limiting form of the 'wave of elevation' is nearly two limiting waves, each with a 'bubble', on either side of the origin. Each wave is nearly a solitary wave by itself, so that $y_0 \approx y_\infty$ and B is small.

This insight shows why α_{max}, which is monotonic, is potentially a more useful parameter than is A or B. However, the corresponding values of α_{max} may not have been determined so accurately as the values of B. We note that the numerical method adopted in LH2 is less suitable for waves of elevation ($B < 0$) because of the slower convergence of the power series, but there seems no reason why the numerical method of VBD should not be used equally well.

Meanwhile, there can be little doubt that the capillary–gravity solitary waves predicted in Longuet-Higgins (1988) and calculated numerically in LH2 and VBD are indeed special, steady, envelope solitons of finite amplitude. These solitons are analytically contiguous to linear capillary–gravity waves, having the minimum phase speed.

A general conclusion to be drawn from the analysis of §3 is that *in any dispersive medium which supports envelope solitons* we may expect the existence of a family of steady solitary waves near a maximum or minimum of the phase speed, that is to say whenever the group velocity nearly equals the phase velocity.

Very recently an analytic study of *steady* solitary waves of small amplitude in finite uniform depth of water has been made by Dias & Iooss (1993). They derive an analytic expression for the surface profile, correct to the third order in a small parameter. In the limiting case of infinite depth the analytic expressions become relatively tractable, and are discussed below in the Appendix. We show there that the amplitude–dispersion relation for their solution, though not derived explicitly by Dias & Iooss (1992) does in fact agree with the dispersion relation which we derived above in equation (3.20), to lowest order. It appears, however, that the higher-order terms given by Dias & Iooss can have at most only a very limited range of validity.

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Appendix. Dias & Iooss (1993)

These authors investigate steady waves on water of finite depth h, characterized by the dimensionless parameters

$$\lambda = gh/c^2, \quad b = T/phc^2 \tag{A 1}$$

so that

$$\lambda b = gT/\rho c^4 \tag{A 2}$$

independently of h. They choose units of length and time so that $T/\rho = 1$, $c = 1$, so in their system $g = \lambda b$ is a variable parameter. They expand the solution about the critical value of λb corresponding to the phase-speed minimum, which in deep water is $\lambda^* b^* = \lambda b = \frac{1}{4}$, as we have seen. Thus they write

$$\lambda b = \tfrac{1}{4} + m, \quad g = \tfrac{1}{4} + m, \tag{A 3}$$

where m is a small parameter.

In the deep-water limit their expression for the surface elevation $\bar{y}$ can be written as

$$\bar{y} = \frac{16n}{11^{\frac{1}{2}}} \frac{\cos \frac{1}{2}\bar{x}}{\cosh n\bar{x}} - \frac{128\, n^2}{11} \frac{\cos \bar{x}}{\cosh^2 n\bar{x}} + \frac{192\, n^3}{11^{\frac{3}{2}}} \frac{\sinh n\bar{x} \sin \frac{1}{2}\bar{x}}{n \cosh^2 n\bar{x}}, \tag{A 4}$$

where $n = m^{\frac{1}{2}}$ and the unit of length is $(T/\rho c^2)$. In the notation of this paper, where $c^2 \doteq 2$ this becomes, to lowest order in n,

$$y = \frac{8n}{11^{\frac{1}{2}}} \frac{\cos x}{\cosh 2nx}. \tag{A 5}$$

Hence we have, again to lowest order

$$\alpha_{\max} \doteq 8n/11^{\frac{1}{2}}. \tag{A 6}$$

We may take n to be positive for waves of depression, negative for waves of elevation.

To obtain the amplitude–dispersion relation note that if we differentiate (A 2) logarithmically we have in general

$$\Delta(\lambda b)/\lambda b = \Delta g/g - 4\, \Delta c/c. \tag{A 7}$$

So in our system, in which $\Delta g = 0$, we have

$$\Delta c/c = -\tfrac{1}{4}\Delta(\lambda b)/\lambda b = -m, \tag{A 8}$$

hence

$$\Delta c = -\surd 2\, n^2. \tag{A 9}$$

On eliminating n between (A 6) and (A 9) we obtain the relation (3.19).

Concerning the range of validity of the expansion (A 4), note that the curvature of the free-surface profile is given, to order n^2, by

$$k = -\frac{4n}{11^{\frac{1}{2}}} + \frac{128 n^2}{11}. \tag{A 10}$$

This changes sign when $n = 11^{\frac{1}{2}}/32$, that is when $\alpha_{\max} \doteq 0.25$, according to (A 6). In the computed profiles of VBD there is no indication of any such reversal of curvature. Thus it appears that the validity of (A 4) is limited to waves of slope small compared to 0.25.

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